Series expansions and sudden singularities

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January 29, 2013

Abstract

We construct solutions of the Friedmann equations near a sudden singularity using generalized series expansions for the scale factor, the density, and the pressure of the fluid content. In this way, we are able to arrive at a solution with a sudden singularity containing two free constants, as required for a general solution of the cosmological equations.

PACS 98.80.Jk.

A sudden singularity will be said to arise everywhere at comoving proper time t_s in a Friedmann universe expanding with scale factor $a(t)$ if

$$\lim_{t\to t_s} a(t) = a_s \neq 0, \qquad \lim_{t\to t_s} \dot{a}(t) = \dot{a}_s < \infty, \qquad \lim_{t\to t_s} \ddot{a}(t) = \infty, \tag{1}$$

for some $t_s > 0$, and we have set $a(t_s) \equiv a_s$. In standard Friedmann-Lemaître cosmological models a fluid is placed in a homogeneous and isotropic spatial geometry whose dynamics is then determined by two independent Einstein equations for three unknown time-dependent functions, the Friedmann metric scale factor, $a(t)$, the fluid density, $\rho(t)$, and fluid pressure, $P(t)$, respectively (with units chosen with $c = 8\pi G = 1$):

$$\frac{\dot{a}^2 + k}{a^2} = \frac{\rho}{3}, \quad -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2 + k}{a^2} = P. \tag{2}$$

When an equation of state $P = f(\rho)$ is chosen this system closes, but we look for solutions of these equations having a sudden singularity, when is no equation of state connecting P and ρ.

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From the Definition (1), it follows that Taylor series are obviously not general enough to describe the local behaviour of solutions near a sudden singularity, and in [1, 2, 3] we began the analysis of such solutions in terms of various types of generalized power series. We use the term *Puiseux series* for an expansion of the form

$$x(t) = \sum_{i=0}^{\infty} a_i \ (t_s - t)^{i/s}; \tag{3}$$

that is, with rational exponents and a constant first term, and we refer to a series development of the form

$$x(t) = (t_s - t)^h \sum_{i=0}^{\infty} a_i \ (t_s - t)^{i/s}, \tag{4}$$

as a *Fuchs series*; that is, when there is an additional real indicial exponent h and no constant term (thus Fuchs and Puiseux series here correspond to Frobenius and Taylor series respectively when $s = 1$, cf. [4]). Setting $x = a, \ y = \dot{a}$, we write the original equations (2) in first-order form,

$$\dot{x} = y, \qquad \dot{y} = -\frac{k + y^2 + x^2 P}{2x}. \tag{5}$$

Below we show that in a dominant solution of this system near a sudden singularity, with leading behaviour $x = a_s$ and $y = \dot{a}_s$ both remaining finite but with the pressure being infinite, all variables of the problem can be expressed in terms of generalized power series with the above forms. In particular, we can show that all solutions of (5) must have the form of Puiseux series

$$x(t) = \sum_{i=0}^{\infty} c_{1i} \ (t_s - t)^{i/s}, \quad y(t) = \sum_{i=0}^{\infty} c_{2i} \ (t_s - t)^{i/s}, \tag{6}$$

where $s > 1$, and

$$c_{10} = \alpha \neq 0, \quad c_{20} = \beta \neq 0, \tag{7}$$

whereas the pressure must be of the form of a Fuchs series,

$$P = (t_s - t)^h \sum_{i=0}^{\infty} p_i \ (t_s - t)^{i/s}, \tag{8}$$

with the restriction that $h < 0$.

By substituting these forms into the field equations, and balancing the various terms, we can determine the series expansions of the scale factor, density and pressure that will lead to a specific sudden singularity. For example, if we choose $n = 3/2$ in the original solution with a sudden singularity of the form [5],

$$a(t) = \left(\frac{t}{t_s}\right)^q (a_s - 1) + 1 - \left(1 - \frac{t}{t_s}\right)^n, \tag{9}$$

then we find that

$$a(t) = \left(\frac{t}{t_s}\right)^q (a_s - 1) + 1 - \left(1 - \frac{t}{t_s}\right)^{3/2} = 1 - \frac{1}{t_s^n}(t_s - t)^{3/2} + \sum_{i=0}^{\infty} a_i (t_s - t)^i \tag{10}$$

$$= (1 + a_0) + a_1(t_s - t) - \frac{1}{t_s^{3/2}}(t_s - t)^{3/2} + a_2(t_s - t)^2 + \cdots = \sum_{i=0}^{\infty} b_{1i}(t_s - t)^{i/2}, \tag{11}$$

where

$$b_{10} = a_s,\, b_{11} = 0,\, b_{12} = a_1,\, b_{13} = -\frac{1}{t_s^{3/2}}, \text{ and for } i \geq 4\, b_{1i} = \begin{cases} 0 & i \text{ odd}, \\ a_{i/2} & i \text{ even}. \end{cases} \tag{12}$$

In this case, the corresponding asymptotic expansions for the pressure and density are given by the following forms (these were first constructed in Ref. [2]):

$$P = \frac{3}{2\alpha t_s^{3/2}}(t_s - t)^{-1/2} - \frac{4\alpha a_2 + \beta^2 + k}{\alpha^2} - \frac{9\beta}{2\alpha^2 t_s^{3/2}}(t_s - t)^{1/2} + \cdots, \tag{13}$$

with

$$a_i = (a_s - 1)\binom{q}{i}\left(-\frac{1}{t_s}\right)^i, \tag{14}$$

and there are *two* independent constants, namely,

$$\alpha = 1 + a_0 = \alpha_s, \text{ (or essentially } t_s) \text{ and } \beta = -a_1, \tag{15}$$

The density has the form

$$\rho = \rho_s + \frac{9\beta}{\alpha^2 t_s^{3/2}}(t_s - t)^{1/2} + \cdots, \text{ with } \rho_s = \frac{3(k + \beta^2)}{\alpha^2}\ . \tag{16}$$

Conversely, starting from the Fuchs expansion

$$P = \sum_{i=0}^{\infty} p_i\ (t_s - t)^{\frac{i}{2} - \frac{1}{2}}, \tag{17}$$

with

$$p_0 = \frac{3}{2\alpha t_s^{3/2}}, \qquad p_1 = -\frac{4\alpha a_2 + \beta^2 + k}{\alpha^2}, \qquad p_2 = -\frac{9\beta}{2\alpha^2 t_s^{3/2}}, \tag{18}$$

and α, β, a_i as above, the system has the following solution for the scale factor,

$$a(t) = \alpha - \beta(t_s - t) - \frac{2\alpha p_0}{3}(t_s - t)^{3/2} - \frac{\alpha^2 p_1 + \beta^2 + k}{4\alpha}(t_s - t)^2 - \frac{6\beta p_0 + 2\alpha p_2}{15}(t_s - t)^{5/2} + \cdots, \tag{19}$$

and the next term will contain p_0, p_1, p_3, and so on; that is, we find the original form,

$$a(t) = \left(\frac{t}{t_s}\right)^q (a_s - 1) + 1 - \left(1 - \frac{t}{t_s}\right)^{3/2}. \tag{20}$$

If in the Fuchs series for the pressure we use a different value of the indicial exponent h, for instance, and we have a total exponent of the form $\frac{i}{2}-1$, or $\frac{i}{2}-\frac{3}{2}$, then the results are essentially the same and the density will be given by:

$$\rho = 3\frac{k+\dot{a}^2(t)}{a^2(t)}. \tag{21}$$

As another example, if

$$P = \sum_{i=0}^{\infty} p_i \, (t_s - t)^{\frac{i}{3}-\frac{2}{3}}, \tag{22}$$

then we find

$$a(t) = \alpha - \beta(t_s - t) - \frac{9\alpha p_0}{8}(t_s - t)^{4/3} - \frac{9\alpha p_1}{20}(t_s - t)^{5/3} + \cdots , \tag{23}$$

with α, β both free constants.

We therefore conclude that with the use of the generalized series methods considered in this paper, the original, 1-parameter solution with a sudden singularity of Ref. [5], becomes part of a two-parameter, general solution of the field equations in the isotropic and homogeneous case with a sudden singularity. A similar result for the general inhomogeneous case has been previously reached in [3], thus enhancing the viability of solutions with sudden singularities.

References

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